Definition (Reflexive Space)

Let X be a normed space and $Q: X \to X^{**}$ be the canonical map, i.e., $Q(x)(x^*) = x^*(x)$. X is called reflexive if Q is surjective.

Rmk: (i) Q is an isometry

(ii) Q is surjective $\Rightarrow X = X^{**}$

(iii) $X = X^{**} \not\Rightarrow Q$ is surjective

Proposition (Three-space property)

Let M be a closed subspace of X. If M and X/M are reflexive, so is X.

Proof: Let $\pi: X \to X/M$ be the natural projection,

$Q_X: X \to X^{**}$, $Q_M: M \to M^{**}$, $Q_{X/M}: X/M \to (X/M)^{**}$

the canonical maps.

Goal: For any $\psi \in X^{**}$, find an $x \in X$ such that $Q_X(x) = \psi$.

Since $\pi^{**}(\psi) \in (X/M)^{**}$ and $Q_{X/M}$ is surjective, there exists some $x_0 \in X$ such that $Q_{X/M}(x_0+M) = \pi^{**}(\psi)$

Then $\pi^{**}(\psi)(\bar{x}^*) = Q_{X/M}(x_0+M)(\bar{x}^*) = \bar{x}^*(x_0+M)$ for all $\bar{x}^* \in (X/M)^*$.

Note that $\pi^{**}(\psi)(\bar{x}^*) = \psi(\pi^*\bar{x}^*) = \psi(\bar{x}^* \circ \pi)$

and $\bar{x}^*(x_0+M) = \bar{x}^*(\pi(x_0)) = \bar{x}^* \circ \pi(x_0) = Q_X(x_0)(\bar{x}^* \circ \pi)$.

Thus $\psi(\bar{x}^* \circ \pi) = Q_X(x_0)(\bar{x}^* \circ \pi)$ for all $\bar{x}^* \in (X/M)^*$.

Then $\psi = Q_X(x_0)$ on $M^\perp$ where

$$M^\perp := \ker(M) = \{x^* \in X^* : x^*(M) = 0\}$$
$$= \{\bar{x}^* \circ \pi : \bar{x}^* \in (X/M)^*\}$$

Therefore, $\psi - Q_X(x_0) = 0$ on $M^\perp$, i.e., $\psi - Q_X(x_0)$ is well-defined on $X^*/M^\perp$.

Recall restriction $\tilde{r}: X^*/M^\perp \to M^*$ is an isomorphic isometry (See Lemma 5.8)

Then $(\psi - Q_X(x_0)) \circ \tilde{r}^{-1} \in M^{**}$.

Since M^* is reflexive, there exists $m_0 \in M$ such that $(\psi - Q_X(x_0)) \circ \tilde{r}^{-1} = Q_{X/M}(m_0)$.

Thus $\psi - Q_X(x_0) = Q_{X/M}(m_0) \circ \tilde{\gamma}$ on $X^*/M^\perp$.

For any $x^* \in X^*$, $x^* + M^\perp \in X^*/M^\perp$.

Then $\psi(x^*) - Q_X(x_0)(x^*) = (\psi - Q_X(x_0))(x^*)$

$$= Q_{X/M}(m_0)(x^*|_M)$$

$$= x^*|_M(m_0)$$

$$= x^*(m_0)$$

$$= Q_X(m_0)(x^*)$$

Therefore, $\psi(x^*) = Q_X(x_0 + m_0)(x^*)$ for all $x^* \in X^*$.

Hence, $\psi = Q_X(x_0 + m_0)$.

$\square$